[3] a, b, c ; 整数

$f(x) = x^3 + ax^2 + bx + c$

$f\left(\frac{1+\sqrt{3}i}{2}\right) = 0$

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(1) $f(x) = x^3 + ax^2 + bx + c$

$$f\left(\frac{1+\sqrt{3}i}{2}\right) = \left(\frac{1+\sqrt{3}i}{2}\right)^3 + a\left(\frac{1+\sqrt{3}i}{2}\right)^2 + b\left(\frac{1+\sqrt{3}i}{2}\right) + c$$

$$= \frac{1+3\sqrt{3}i+9i^2+3\sqrt{3}i^3}{8} + a\,\frac{1+2\sqrt{3}i+3i^2}{4} + b\,\frac{1+\sqrt{3}i}{2} + c$$

$$= \left(-1 - \frac{a}{2} + \frac{b}{2} + c\right) + \left(\frac{\sqrt{3}}{2}a + \frac{\sqrt{3}}{2}b\right)i = 0$$

a, b は整数より, $-1 - \frac{a}{2} + \frac{b}{2} + c$ と $\frac{\sqrt{3}}{2}a + \frac{\sqrt{3}}{2}b$ が整数

$$\begin{cases} -1 - \frac{a}{2} + \frac{b}{2} + c = 0 \\ \frac{\sqrt{3}}{2}a + \frac{\sqrt{3}}{2}b = 0 \end{cases} \Leftrightarrow \begin{cases} a = c-1 \\ b = -c+1 \end{cases}$$

(2) 題意より,

$f(1) = 7k + 4$

$f(-1) = 11\ell + 2$ (k, ℓ は整数)

(1)より $f(x) = x^3 + (c-1)x^2 + (-c+1)x + c$

$f(1) = 1 + c - 1 - c + 1 + c$

$= c + 1 = 7k + 4$

$c = 7k + 3$

$f(-1) = -1 + c - 1 + c - 1 + c$

$= 3c - 3 = 11\ell + 2$

$3c = 11\ell + 5$

$c = \frac{11\ell + 5}{3}$

$|c| \leqq 40$ より, $|7k+3| \leqq 40$ 整数

$k = -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$ (12個)

$\left|\frac{11\ell+5}{3}\right| \leqq 40$ 整数

$\ell = -10, -7, -4, -2, 5, 8$ (6個)

$-35, -22, -13, 9, 20$

(1)の時に満たすのは $c = 3\ell$ のとき

$a = 30$

$b = -30$

$f(x) = x^3 + 30x^2 - 30x + 31$

$x = \frac{1+\sqrt{3}i}{2}$ を解にもつので

$2x = 1 + \sqrt{3}i$

$2x - 1 = \sqrt{3}i$

$4x^2 - 4x + 1 = -3$

$4x^2 - 4x + 4 = 0$

$x^2 - x + 1 = 0$

$x^3 + 30x^2 - 30x + 31 = (x+31)(x^2 - x + 1) = 0$

$x = -31, \frac{1 \pm \sqrt{3}i}{2}$ (3個)

$\therefore -31, \frac{1+\sqrt{3}i}{2}$